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SEM – V (Paper I)

TOPIC:- ANALYTIC FUNCTION

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Definition of Analytic function

A function f(z) is said to be analytic at z_0 if it is differential at z and in some neighbourhood of z_0. A function f(z) is analytic on a region D if its derivative f'(z) exists at all point z ϵ D.

CAUCHY- RIEMANN EQUATION

Statement

A necessary condition that

$f(z) = u(x,y) + iv(x,y)$

be analytic in a region D is that

$u_x = v_y$ and $u_y = -v_x$ in D

Proof:- Let $f(z) = u(x,y) + iv(x,y)$ ....(1) be analytic in region D.

Then $f(z)$ is differentiable at all point of D. i.e. $f'(z)$ exists $\forall\, z \in D$.

By definition, $$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

Since, $\delta z = \delta x + i\delta y$, from eq^n (1) we write,

$$f'(z) = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{\{u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)\} - \{u(x,y) + iv(x,y)\}}{\delta x + i\delta y}$$

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta x + i\delta y} + i \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta x + i\delta y} \quad \text{.........................(2)}$$

We compute f'(z) in (2)in two different ways by taking two different paths.

CASE (I). Choose, $\delta z = \text{real} = \delta x$ i.e. $\delta y = 0$. Then (2) becomes

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta X, y) - u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

$$= u_x + iv_x \quad \ldots\ldots\ldots\ldots(3a)$$

CASE (II). Choose δz = imaginary = $i\delta y$ i.e. δx=0. Then (2) becomes

$$f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{i\delta y}$$

$$= \frac{1}{i} u_y + v_y$$

$$= -iu_y + v_y \quad \ldots\ldots\ldots\ldots(3b)$$

By uniqueness of f'(z), (3) becomes, $u_x + iv_x = -iu_y + v_y$

Equating real and imaginary parts,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

Hence proved.

APPLICATION:- Show that the C-R equation are satisfied only at $z = 0$ for $f(z) = |z|^2$

Proof:- Here $f(z) = u + iv = |z|^2 = |x + iy|^2 = x^2 + y^2$.

Equating real and imaginary parts, $u = x^2 + y^2$, $v = 0$

It becomes, $u_x = 2x$, $u_y = 2y$, $v_x = 0$, $v_y = 0$.

The C-R Eq^n, $u_x = v_y$, $u_y = -v_x$ Becomes, $2x = 0$, $2y = 0$ i.e. $x = 0$, $y = 0$

Becomes, $z = x + iy = 0$

At any $x \neq 0$, $u_x \neq v_y$ and at any $y \neq 0$, $u_y \neq -v_x$

i.e., the C-R eq^n are satiafied only at $z = 0$.

THANK YOU